

On cycles through two arcs in strong multipartite tournaments

Alexandru I. Tomescu*

Abstract

A multipartite tournament is an orientation of a complete c -partite graph. In [L. Volkmann, A remark on cycles through an arc in strongly connected multipartite tournaments, Appl. Math. Lett. 20 (2007) 1148–1150], Volkmann proved that a strongly connected c -partite tournament with $c \geq 3$ contains an arc that belongs to a directed cycle of length m for every $m \in \{3, 4, \dots, c\}$. He also conjectured the existence of three arcs with this property. In this note, we prove the existence of two such arcs.

Keywords: multipartite tournament; cycle; cycle through an arc.

A c -partite or *multipartite tournament* is an orientation of a complete c -partite graph. By a *cycle* or *path* we mean a simple directed cycle or simple directed path. For standard terminology on directed graphs see, e.g., Bang-Jensen and Gutin [1]. In what follows, all digraphs are finite, without loops or multiple arcs. The vertex set of a digraph D is denoted by $V(D)$. If xy is an arc of a digraph D , then we write $x \rightarrow y$ and say that x dominates y . If X and Y are two disjoint subsets of $V(D)$, such that every vertex of X dominates every vertex of Y , then we say that X dominates Y , and write $X \rightarrow Y$. A digraph D is *strongly connected* or *strong* if for each pair of vertices u and v there is a path in D from u to v . A cycle of length m is also called an m -cycle. A digraph D is *pancyclic* if it contains an m -cycle for all m between 3 and $|V(D)|$. A vertex or an arc is *pancyclic* in a digraph D if it belongs to an m -cycle for all m between 3 and $|V(D)|$.

Moon [2] obtained the following result on pancyclic arcs for strongly connected tournaments.

Theorem 1 (Moon [2]) *Every strong tournament contains at least three pancyclic arcs.*

In [3], Volkmann showed that a similar result holds for the case of strong multipartite tournaments.

Theorem 2 (Volkmann [3]) *If D is a strong c -partite tournament with $c \geq 3$, then D contains at least one arc that belongs to an m -cycle C_m for each $m \in \{3, 4, \dots, c\}$ such that $V(C_3) \subset V(C_4) \subset \dots \subset V(C_c)$.*

Volkmann also proposed [3, 4] a possible improvement of this result, by formulating the following conjecture.

*Dipartimento di Matematica e Informatica, Università di Udine, Via delle Scienze, 206, 33100 Udine, Italy, alexandru.tomescu@uniud.it

Faculty of Mathematics and Computer Science, University of Bucharest, Str. Academiei, 14, 010014 Bucharest, Romania

Conjecture 1 (Volkman [3]) *If D is a strong c -partite tournament with $c \geq 3$, then D contains at least three arcs that belong to an m -cycle for each $m \in \{3, 4, \dots, c\}$.*

In this note we will prove a weaker version of this conjecture, by showing that in a c -partite strong tournament there are two arcs that belong to an m -cycle for each $m \in \{3, 4, \dots, c\}$. Note that the proof of Theorem 3 below works along the same lines as the proof of Theorem 2.

Theorem 3 *If D is a strong c -partite tournament with $c \geq 3$, then D contains two arcs e^1, e^2 , where each $e^k \in \{e^1, e^2\}$ belongs to an m -cycle C_m^k , for each $m \in \{3, 4, \dots, c\}$, such that $V(C_3^k) \subset V(C_4^k) \subset \dots \subset V(C_c^k)$.*

Proof: It is known that D contains a 3-cycle (see, for example, [3]). Any two arcs of this cycle satisfy the claim for $m = 3$.

Suppose now that for some m satisfying $3 \leq m < c$, there are two arcs e^1 and e^2 , such that each $e^k \in \{e^1, e^2\}$ belongs to $m - 2$ cycles $C_3^k, C_4^k, \dots, C_m^k$ with: $|V(C_j^k)| = j$, for all $3 \leq j \leq m$, and $V(C_3^k) \subset V(C_4^k) \subset \dots \subset V(C_m^k)$.

In what follows, we will show that for each $k \in \{1, 2\}$ there are two ways to satisfy the inductive requirement. First, e^k can also be contained in a cycle C_{m+1}^k of length $m + 1$, such that $V(C_m^k) \subset V(C_{m+1}^k)$. Second, we can find two arcs f^1, f^2 , belonging to a j -cycle for each $3 \leq j \leq m + 1$, with the desired properties.

Let $e \in \{e^1, e^2\}$ and let $C_m = u_1 u_2 \dots u_m u_1$, where $e = u_1 u_2$, be the cycle of length m obtained from the inductive hypothesis. Let S be the set of vertices that belong to partite sets not represented on C_m .

If S contains a vertex w that has an out-neighbor and an in-neighbor on C_m , then w has an in-neighbor u^- on C_m immediately followed by an out-neighbor u^+ on C_m . If $u^- \neq u_1$ and $u^+ \neq u_2$, then we can form an $(m + 1)$ -cycle C_{m+1} containing e by replacing the arc $u^- u^+$ of C_m with the arcs $u^- w, w u^+$. Observe that $V(C_m) \subset V(C_{m+1})$.

Assume now that $u^- = u_1$ and $u^+ = u_2$, and there are no further two vertices on C_m with these properties. Observe first that there exists an index $2 \leq i \leq m$ such that $w \rightarrow \{u_2, u_3, \dots, u_i\}$ and $\{u_{i+1}, u_{i+2}, \dots, u_m, u_1\} \rightarrow w$. The arc $u_i u_{i+1}$ is contained in cycles of lengths $3, 4, \dots, m + 1$ with the desired properties, because $u_i u_{i+1} \dots u_{i+j} w u_i$, where $u_{m+1} = u_1$ are cycles of lengths $3, 4, \dots, m + 3 - i$ containing $u_i u_{i+1}$, for $1 \leq j \leq m + 1 - i$ and $u_{i-k} u_{i+1-k} \dots u_i u_{i+1} \dots u_m u_1 w u_{i-k}$ are cycles of length $m + 4 - i, m + 5 - i, \dots, m + 1$ containing $u_i u_{i+1}$ for $1 \leq k \leq i - 2$, when $i \geq 3$.

Second, as $u_1 u_2$ belongs to a 3-cycle, there exists a vertex $v \in V(D)$ such that $u_1 u_2 v u_1$ is a cycle. From the inductive hypothesis, we also have $v \in V(C_m)$. As v belongs to a different partite set than w , we have either $w \rightarrow v$, or $v \rightarrow w$. In the former case, the arc $u_1 w$ belongs to the 3-cycle $C'_3 = u_1 w v u_1$, while in the latter case, the arc $w u_2$ belongs to the 3-cycle $C'_3 = w u_2 v w$. Additionally, both $u_1 w$ and $w u_2$ are contained in the 4-cycle $C'_4 = u_1 w u_2 v u_1$. In general, as $V(C_3) \subset V(C_4) \subset \dots \subset V(C_m)$ and the fact that w belongs to a partite set not represented on C_m , every cycle C_i of length i , $4 \leq i \leq m$, containing $u_1 u_2$ can be extended to a cycle C'_{i+1} of length $i + 1$, by replacing the arc $u_1 u_2$ with the two arcs $u_1 w$ and $w u_2$. Moreover, we have $V(C'_3) \subset V(C'_4) \subset \dots \subset V(C'_{m+1})$. In conclusion, we can replace the arcs e^1, e^2 by the arc $f^1 = u_i u_{i+1}$ and an arc $f^2 \in \{u_1 w, w u_2\}$, $f^2 \neq f^1$, and satisfy our claim.

The only case it remains to consider is when S can be decomposed into two subsets S_1 and S_2 such that $S_2 \rightarrow V(C_m) \rightarrow S_1$. Since $m < c$, we may assume, without loss of generality, that $S_1 \neq \emptyset$. As D is strongly connected, there is a path from S_1 to C_m . Let $P = y_1 y_2 \dots y_q$ be a shortest such path, where $q \geq 3$, $y_1 \in S_1$ and $y_q \in V(C_m)$. We note that $y_t \notin S_1 \cup S_2$, for all $2 \leq t \leq q - 2$, as otherwise the minimality of P would be contradicted. Therefore, we also have $y_t \rightarrow y_1$, for all $2 \leq t \leq q - 2$.

When $y_{q-1} \in S_2$, we have an arc between y_{q-1} and every vertex on C_m . Let u_z be a vertex of C_m such that u_z and y_{q-2} are in different partite sets. This entails $u_z \rightarrow y_{q-2}$. The arcs $f^1 = u_z y_{q-2}$ and $f^2 = y_{q-2} y_{q-1}$ are both contained in cycles of length $3, 4, \dots, m+1$ because $u_z y_{q-2} y_{q-1} u_j u_{j+1} \dots u_{z-1} u_z$ are cycles of length $m+3-j$, for $2 \leq j \leq m$, with the desired property.

Consider now $y_{q-1} \notin S_2$. If $q > 3$, the minimality of P implies that y_{q-1} and y_1 belong to different partite sets, and hence $y_{q-1} \rightarrow y_1$. If $q = 3$ then $y_3 \in C_m$, and since $y_1 \in S_1$ and $V(C_m) \rightarrow S_1$, we have $y_3 \rightarrow y_1$. Therefore, the arcs $f^1 = y_1 y_2$ and $f^2 = y_2 y_3$ are both contained in a j -cycle, for $3 \leq j \leq q$. Since $V(C_m) \rightarrow S_1$, we deduce that they are also contained in a j -cycle for $q+1 \leq j \leq m+q-1$. In all, we see that both f^1 and f^2 belong to a j -cycle C'_j for $j = 3, 4, \dots, m+q-1$, with $m+q-1 \geq m+2$, such that $V(C'_3) \subset V(C'_4) \subset \dots \subset V(C'_{m+1})$. \square

References

- [1] J. Bang-Jensen, G. Gutin, *Digraphs: Theory, Algorithms and Applications*, Springer, London, 2000.
- [2] J.W. Moon, *On k -cyclic and pancyclic arcs in strong tournaments*, J. Combin. Inform. System Sci. **19** (1994) 207–214.
- [3] L. Volkmann, *A remark on cycles through an arc in strongly connected multipartite tournaments*, Appl. Math. Lett. **20** (2007) 1148–1150.
- [4] L. Volkmann, *Multipartite tournaments: A survey*, Discrete Math. **307** (2007) 3097–3129.